Question 1

1. *Which statement below is true about the series* $\sum_{n=1}^{\infty} \frac{e^n}{n^2+e^n}$ *?*

$\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 1$ so the series diverges.

$\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 1$ so the series converges.

$\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 0$ so the series diverges.

$\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 0$ so the series converges.

$\lim_{n\to\infty} \frac{e^n}{n^2+e^n}$ does not exist so the series converges.

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$\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 0$ so the series converges.
$\lim_{n\to\infty} \frac{e^n}{n^2+e^n}$ does not exist so the series converges.

- $\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 1$ so the series diverges.

Question 2

2. The series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

a. does not converge absolutely but does converge conditionally.

b. converges absolutely.

c. diverges because the terms alternate.

d. diverges because $\lim_{n\to\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \neq 0$.

e. diverges even though $\lim_{n\to\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 0$.

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- This is an alternating series of type $\sum (-1)^{n+1} b_n$, with $b_n > 0$.

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- This is an alternating series of type $\sum(-1)^{n+1}b_n$, with $b_n > 0$.
- We apply the alternating series test: $(ii)\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/\sqrt{n} = 0$

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- (i) The sequence $\{b_n\}_{n=2}^{\infty}$ is decreasing since $\sqrt{n+1} > \sqrt{n}$ and thus $b_{n+1} = 1/\sqrt{n+1} < 1/\sqrt{n} = b_n$ for all $n \geq 2$.

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- Therefore the series $\sum (-1)^{n+1} b_n$ converges, ruling out 3 of the answers.

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- Therefore the series $\sum(-1)^{n+1} b_n$ converges, ruling out 3 of the answers.
- The series $\sum_{n=2}^{\infty} |\frac{(-1)^{n+1}}{\sqrt{n}}| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it's a p series and $p = \frac{1}{2} < 1$. Therefore the series does not converge absolutely.

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- Therefore the series $\sum(-1)^{n+1}b_n$ converges, ruling out 3 of the answers.
- The series $\sum_{n=2}^{\infty} |\frac{(-1)^{n+1}}{\sqrt{n}}| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it's a p series and $p = \frac{1}{2} < 1$. Therefore the series does not converge absolutely.
- Therefore The series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ does not converge absolutely but does converge conditionally.

Question 3

3. *Use Comparison Tests to determine which* **one** *of the following series is divergent.*

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- (a) $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, a p-series with $p = \frac{3}{2} > 1$.

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- (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+8}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series with $p = 2 > 1$.

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- (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+8}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series with $p = 2 > 1$.
- (c) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, a p-series with $p = 1$.

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- (c) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, a p-series with $p = 1$.
- (d) $\sum_{n=1}^{\infty} 7\left(\frac{5}{6}\right)^n$ converges since it is a geometric series with $|r| = \frac{5}{6} < 1$.

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- (e) $\sum_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{1}{2}\right)^n$ converges by comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, a geometric series with $|r| = \frac{1}{2} < 1$.

Question 4

4. Consider the following series

$$(I) \quad \sum_{n=1}^{\infty} \left(\frac{2n^2+7}{n^2+1} \right)^n \qquad (II) \quad \sum_{n=2}^{\infty} \frac{2^{1/n}}{n-1} \qquad (III) \quad \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

Which of the following statements is true?

They all diverge.

(I) converges, (II) diverges, and (III) diverges.

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- For (I), we apply the n th root test. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n^2+7}{n^2+1}$ $= \lim_{n\to\infty} \frac{2+7/n^2}{1+1/n^2} = 2 > 1$. Therefore the series diverges.

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- $\sum_{n=2}^{\infty} \frac{2^{1/n}}{n-1}$ diverges by direct comparison with the series $\sum \frac{1}{n}$, since $\frac{2^{1/n}}{n-1} > \frac{1}{n-1} > \frac{1}{n}$ for all n.

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- For III, we apply the ratio test, $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(n+1)!}{e^{n+1}}\Big/\frac{n!}{e^n}$ $= \lim_{n\to\infty}\frac{n+1}{e} = \infty > 1$. Therefore the series diverges.

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- $\sum_{n=2}^{\infty}\frac{2^{1/n}}{n-1}$ diverges by direct comparison with the series $\sum\frac{1}{n}$, since $\frac{2^{1/n}}{n-1} > \frac{1}{n-1} > \frac{1}{n}$ for all n.
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- Therefore they all diverge.

Question 5

5. *Which series below conditionally converges?*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{n}}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$$

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- Recall that a series is conditionally convergent if it is convergent but not absolutely convergent.

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^n}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$$

- Recall that a series is conditionally convergent if it is convergent but not absolutely convergent.
- Note immediately that $\sum_{n=1}^{\infty}(-1)^{n-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^n}{\sqrt{n}}$ are divergent as their terms tend not to zero as n goes to infinity.

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^n}{\sqrt{n}} \qquad \sum_{n=1}^{\infty}(-1)^{n-1} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$$

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- Now, the other series are convergent by the alternating series test.

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^n}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$$

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- Note immediately that $\sum_{n=1}^{\infty} (-1)^{n-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^n}{\sqrt{n}}$ are divergent as their terms tend not to zero as n goes to infinity.
- Now, the other series are convergent by the alternating series test.
- Further, considering the corresponding series given by taking the absolute value term wise we see that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ are absolutely convergent , while $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is not.

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- Recall that a series is conditionally convergent if it is convergent but not absolutely convergent.
- Note immediately that $\sum_{n=1}^{\infty} (-1)^{n-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^n}{\sqrt{n}}$ are divergent as their terms tend not to zero as n goes to infinity.
- Now, the other series are convergent by the alternating series test.
- Further, considering the corresponding series given by taking the absolute value term wise we see that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ are absolutely convergent , while $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is not.
- Hence $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ alone is conditionally convergent.

Question 6

6. Consider the following series

$$(I) \quad \sum_{n=1}^{\infty} \frac{2^n}{n!} \qquad\qquad (II) \quad \sum_{n=1}^{\infty} \left(\frac{n^2+n}{2n^2+1}\right)^n$$

Which of the following statements is true?

(a) They both converge. (b) They both diverge. (c) (I) converges and (II) diverges. (d) (I) diverges and (II) converges. (e) The Ratio Test applied to (I) is inconclusive.

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- Both series converge, to see this we apply the ratio test to series (I) and the root test to series (II). Indeed, let $a_n = \frac{2^n}{n!}$ and $b_n = \left(\frac{n^2+n}{2n^2+1}\right)^n$ then

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- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

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- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$
- and $\lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \frac{n^2+n}{2n^2+1} = \frac{1}{2} < 1$

Question 6

6. *Which series below is the MacLaurin series (Taylor series centered at 0) for*

$$\frac{x^2}{1+x}?$$

a. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\, x^{n+2}$ b. $\displaystyle\sum_{n=0}^{\infty} x^{2n+2}$ c. $\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}$ d. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n x^{2n-2}}{n!}$

e. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$

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e. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$

- $\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^{n+2}$, for $|x| < 1$.

Question 7

7. *Which series below is a power series for* $\cos(\sqrt{x})$ *?*

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$$\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{x}^{\,n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-\frac{1}{2}}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^2+1}$$

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- $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

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$$\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9}.$$

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$$\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots,$$

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- and

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right) - x^3}{x^9}$$

$$= \lim_{x\to 0} \frac{-\frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots}{x^9} = -\frac{1}{6}.$$

Question 9

9. The following is the fifth order Taylor polynomial of the function $f(x)$ at a

$$T_5(x) = 2 - 2(x-a) + \sqrt{5}(x-a)^2 - \frac{\pi}{2}(x-a)^3 + (x-a)^4 + 13(x-a)^5$$

What is $f^{(3)}(a)$?

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- Therefore
$$f^{(3)}(a) = -3\pi.$$

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10. *Does the series*

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$$

converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

Solution 1:

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- Since the limit is > 1, the series diverges.

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111. *Find the radius of convergence and interval of convergence of the power series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x-3)^n$

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- Set $a_n = \frac{(-1)^n}{\sqrt{n}}(x-3)^n$. Using the Ratio Text,

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- Hence, the radius of convergence is 1, and the series converges absolutely for $|x-3| < 1$, or $2 < x < 4$.
- For the end points, when $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent since it is a p-series with $p = \frac{1}{2} < 1$.

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- when $x = 4$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n}}$ which is convergent since it's an alternating series, and $b_n = \frac{1}{\sqrt{n}}$ is decreasing and $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. (See the solution to Problem #3 for details.)

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- Hence, the interval of convergence is $2 < x \leq 4$.

Question 12

12. *(a) Show that* $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$ *provided that* $|x| < 1$.

(b) Find $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}$.

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- Therefore $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C$.

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- Letting $x = 0$, we have $C = 0$.

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- Hence, we have $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x$. Let $x = \dfrac{1}{\sqrt{3}}$.
- We get $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$.